Solid State II

Homework for Week 8
Due Thursday, 27 October, 2005

Worked problems to prepare: Blundell 6.1, 6.6, 6.7
To turn in: Blundell 6.9 and

1. The Heisenberg model is approximated by a large number of related models based on particles instead of spin; examples include the Bosonic Holstein-Primakov and Schwinger models. This problems works out instead an exact mapping: the one-dimensional Heisenberg model

\[
\mathcal{H} = \sum_j (S_j^z S_{j+1}^z + \frac{1}{2}[S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+])
\]

on an infinite lattice maps exactly to the spinless tight-binding-Fermion model from the first homework, plus an additional nearest-neighbor interaction term, plus a constant.

a. Show that the substitutions

\[
S_j^+ = (S_j^-)\dagger = c_j \exp\left(i\pi \sum_{r<j} c_r^{\dagger} c_r\right)
\]

\[
S_j^- = c_j^{\dagger} c_j - 1/2
\]

written in terms of Fermionic operators satisfying anticommutation relations \(\{c_i^{\dagger}, c_j\} = c_i^{\dagger} c_j + c_j c_i^{\dagger} = \delta_{i,j}, \quad \{c_i, c_j^{\dagger}\} = \{c_i, c_j\} = 0\) yield the correct commutation algebra for the spin operators on the same site. As a peculiar but ultimately unimportant artifact, show that spin operators on different sites anticommute in this model instead of commuting.

b. Remember the meaning of the operator \(c_i^{\dagger} c_j\).

c. Show that

\[
\exp(i\pi c_j^{\dagger} c_j) = 1 - 2c_j^{\dagger} c_j = (c_j^{\dagger} + c_j)(c_j^{\dagger} - c_j)
\]

and that the square of this is unity.

d. Show that

\[
c_j^{\dagger} \exp(i\pi c_j^{\dagger} c_j) = c_j^{\dagger} \]

\[
c_j \exp(i\pi c_j^{\dagger} c_j) = -c_j
\]

e. Because \(\mathcal{H}\) has only first-neighbor spin interactions, the nasty exponentials of part (a) nearly cancel in the \(S_j^+ S_{j+1}^-\) and \(S_j^- S_{j+1}^+\) terms. Make the substitutions to arrive at the tight-binding model described. Note that particle number is preserved, so any term that just counts a total number of particles in the system may be treated as a constant.

f. What is the relation between the total number of particles in the Fermionic system and the total spin (\(S^z\) eigenvalue) of the spin problem?

The Fermionic interaction term (if you haven’t got this yet, review part (b)) makes clear that this (and therefore also the original Hamiltonian) is a many-body problem: it cannot be understood simply by filling up single-body states to a Fermi level.