Things do mix even if energetically disfavored: entropy of mixing.

$A, B$ noteds  

(PV negligible, so well minimize F interact.)

homogeneous $A_x B_{1-x}$ or phase separation

**PHASE SEPARATED first:**

If $A, B$ have two bulk states, $S = S_0 = (1-x)S_A + xS_B$, present whether phase segregated or not.

Can its structure differ the entropies will be comparable - we'll use equal.

Entropy of mixing in homogeneous mixture by

$x_{A_v} = \frac{x}{N_A + x}$

$x_{B_v} = \frac{N_B}{N_A + x}$

\[ m = \left( \frac{V_A}{V_B} \right) = \frac{V_A}{V_B} \frac{V_C}{V_C} = \frac{V_A}{(1-x)V_A + xV_B} \]

\[ \rho = \frac{k_B}{g} \left[ \ln \frac{V_A}{V_B} - 1 \right] \]

\[ \frac{1}{N_A} \ln \frac{V_A}{V_B} \]

\[ \rho = \frac{k_B}{g} \ln \frac{V_A}{V_B} \]

\[ \rho = \frac{k_B}{g} \ln \frac{V_A}{V_B} \]

This term exist in phase-separated configuration.

\[ \rho = \frac{S}{N} \quad \rho_A = \frac{S_A}{N} \]

\[ \rho_B = \frac{S_B}{N} \]

\[ \rho_A = \frac{S_A}{N} \]

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\[ \rho_B = \frac{S_B}{N} \]

\[ \rho = \frac{S}{N} \]

\[ \rho_A = \frac{S_A}{N} \]
At high enough \( T \), it is dominated by \( -\Gamma_{\Delta m}^2 \).

Hypothetical mixture (---) line, lower than ideal: this phase segregated easily; - - for any \( x \).

So A \& B are miscible at this temp.

If low temp, \( \beta_x \) is still important at edges because of \( \infty \) slope:

\[
\frac{\partial \beta_x}{\partial x} \bigg|_{x=1} = \lim_{x \to 1} \frac{-\left( V_b \Gamma \int_x^{1-x} (1-x) \ \beta_x \right)}{x^{1/3}} = \lim_{x \to 1} \left( V_b \Gamma \int_x^{1-x} (1-x) \ \beta_x \right) \frac{1}{x^{1/3}} = 0
\]

For \( x < x_1 \), hypothetical mixture \( A_1 + B_x \) has lowest \( T \), similar to \( A_2 \times x_1 \). However, between \( x_1 \& x_2 \), there is no mixture into the pure.

\[
\begin{align*}
\Delta \text{mix} & = \Gamma_{\Delta m}^2 \left( \frac{x_1}{x_2} \right) \\
\Delta \text{mix} & = \Gamma_{\Delta m}^2 \left( \frac{x_2}{x_1} \right)
\end{align*}
\]

\( x_1 \& x_2 \) are NOT minima, other, tangent points.
$\Psi_263$  Mono: entropy of mixing phase separation

today: phase separation extistics
stark cryogenics
Fri: cryogenics

[last two graphs]

$^3$He - $^4$He mixture has similar solubility gap but for a different reason.
I don’t think there’s much in the way of $^3$He-$^4$He energy, much less a difference among $\lambda_{33}$, $\lambda_{44}$, and $\lambda_{43}$. Supersolid $^3$He at $13 \text{ mK}$ ($T \approx T_A$) contains almost no energy, so sort of $^4$He:

$\text{Fermions } \frac{r^2}{V} = \frac{E_F}{\frac{3}{2}} \Rightarrow \frac{E_F}{r^2} \approx \frac{3}{2} E_{F}E_{F} = E_{F}\sqrt{3}$

$E_F \sim n^{5/3}, \text{ so } \frac{U_0}{V} \sim n^{5/3}, \text{ or } \frac{U_0}{N} \sim \frac{V}{N}n^{5/3} = \left(\frac{V}{N}\right)^{5/3}$
Mixture does not solidify at one temp, but over a range: when hit liquidus, T

A precipitate forms not at \( x_e \) but at \( x_s \), depleting the B component and leaving the liquid more A-rich, so it solidifies at a lower temperature. Solidification is not complete until \( T_f \).

How can a binary mixture have a melting temp. that is lower than that of either of its constituents?

As Si system - differs from Al-Si in having a solubility gap in the solid only.

Eutectic mixture:
At \( T_e \), \( x_c \), \( y_{xe} = y_{ce} \)
If we start w/ liquid \( y_c \) above \( T_e \), it solidifies at the lower solidus \( T_s \) of the pure material.
At any time \( x_e \) (\( p.x \)), it solidifies...
Practical cryogenics

Down to (cryo-)15K? - closed cycle refrigeration

77K: boiling pt of liquid N₂ at room pressure

5K: boiling at 7T (T=4.2K)

Can get down to -1K by evaporative cooling

Called "pumping and"

Maybe increase the (4-5K) pumping on 7T, but very expensive or need closed cycle

Infiltration: dilution refrigerators

Low-ε: Tomasch's refrigeration

Millimprots = laser cooling

Show Jeff Olson's paper on low-T thermal conductivity with spires

4/40 Wed.: Categorical cryogenics

Today: Cryogenics

Josh expansion

Just-finished dilution cryostat

Heat: laser-cooling

Show simplified schematic of cryostat

(some recent design omit N₂ stage)
Reversible refrigeration becomes difficult at low temp., particularly when the working fluid would liquify during the adiabatic-expansion stage, so for practical reasons, we're drawn to irreversible processes.

Joule effect - sudden (not isostropic) expansion, revisited

![Diagram of Joule effect]

Irreversible. Recall for ideal gas, no change in temp. since $U = \frac{2}{3}Nk_B \ln(T)$ only.

More generally, $U = U(T, V)$.

We can still use equilibrium thermodynamics, as the initial and final states are (separately) equilibria.

$$\begin{align*}
\Delta U &= 0 = \left( \frac{\partial U}{\partial T} \right)_V dT|_V + \left( \frac{\partial U}{\partial V} \right)_T dV|_T \quad \text{indicate constant } U \\
\Rightarrow \quad \left( \frac{\partial T}{\partial V} \right)_U &= -\frac{\left( \frac{\partial U}{\partial V} \right)_T}{\left( \frac{\partial U}{\partial T} \right)_V} \\
\end{align*}$$

denominator: $\left( \frac{\partial U}{\partial T} \right)_V = C_V$, so

$$\left( \frac{\partial T}{\partial V} \right)_U = -\left( \frac{\partial U}{\partial V} \right)_T / C_V$$

- Numerator always positive to $T_{eq}$:

![Graph of temperature change]

so $\left( \frac{\partial T}{\partial V} \right)_U < 0 \quad \Rightarrow \quad$ expansion cools, an interesting way.

Evaluate numerator: Recall from ch. 3 (Since) (lectur- p25, 11/24/03)

$$P = -\left( \frac{\partial U}{\partial V} \right)_T + T \left( \frac{\partial S}{\partial V} \right)_T$$

so numerator: $\left( \frac{\partial U}{\partial V} \right)_T = P - T \left( \frac{\partial S}{\partial V} \right)_T$
Thus \[
\left( \frac{\partial T}{\partial V} \right)_U = \frac{1}{c_v} \left[ P - T \left( \frac{\partial P}{\partial T} \right)_V \right].
\]

Ideal gas: \[ P = \frac{N k_b T}{V}, \] so \[ T \left( \frac{\partial P}{\partial T} \right)_V = P \Rightarrow \left( \frac{\partial T}{\partial V} \right)_U = 0. \]

(NaA) ideal gas:

\[ P = \frac{N k_b T}{V} \left[ 1 + \frac{B_2(T)}{V^2} + \frac{B_3(T)}{V^3} + \ldots \right] \]

called virial or

Keesom-Mayer-Teller expansion

\[ B_2 = B_3 = \ldots = 0 \] for ideal gas

Can calculate \( B_2(T) \) for non-ideal gases - examples: van der Waals, Bose, Fermi

\( P(T, V) \) is a Laurent rather than a Taylor expansion, because it blows up at \( V = 0 \).

If \( \frac{B_2}{V^2}, \ldots \) are negligible (expected at low density, \( \frac{N}{V} \ll n_o \)),

\[ T \left( \frac{\partial P}{\partial T} \right)_V = \frac{N k_b T}{V} \left[ 1 + \frac{B_2(T)}{V^2} \frac{d}{dT} \right] = P + N k_b T \frac{\partial B_2}{\partial T}
\]

so \[ \left( \frac{\partial T}{\partial V} \right)_U = -\frac{1}{c_v} \frac{N k_b T^2}{V^2} \frac{\partial B_2}{\partial T} \]

called "Sund" effect.

Similar, more useful:

\[ P_2 \rightarrow \rightarrow P_2 < P_1 \]

equivalently, pores, realistic partition

definitive, reversible, but non
we call this pressure instead of volume in suitable
for vacuum, porous
at liquefaction
\[ 
\Delta U = U_2 - U_1 = P_2 V_2 - P_1 V_1 \\
U_1 + P_1 V_1 = U_2 + P_2 V_2 \\
\text{entropies equal: } H_1 = H_2
\]

Intuition:
\[
V \leftrightarrow P, \quad \text{constant } U \rightarrow \text{constant } H
\]

So guess:
\[
\left( \frac{2T}{V^2} \right)_H = -\frac{1}{\gamma_p} \left[ V - T \left( \frac{\partial V}{\partial P} \right)_T \right]
\]

Good guess except sign (so < sign)

called Joule - Thomson effect (Thomson = Lord Kelvin)

Unlike Joule expansion, sign depends on T.
For any gas, there's an inversion temperature below which this cools, above which it heats, so in limitation it's first necessary to cool below inversion temperature.

Intuition: Consider:

Fundamental idea: The gas is just transferred adiabatically into the tank evaporating into the vacuum.

"Evaporation is done, so up"

Go through phase 1, then...
Historically important cooling method: Pennington effect

Used by Lee, Conrath, Richardson in 1912 to freeze superfluid $^4$He.

Refer to Bob Richardson in his 1996 Nobel lecture, RMP 683 (1997).

```
compass solidifying $^4$He

\[ \text{precipitate } ^3\text{He} \]
```

Entropy of solid $>\text{entropy of liquid}$, so it takes heat to solidify $\rightarrow$ Liquid.

Equal normal $^3$He + position liquid.

So only a small proportion of states are available for entropy.

Classical - Clausius-Clapeyron:
\[
\frac{dP}{dT} = \frac{\Delta_{\text{heat}} - \Delta_{\text{work}}}{\Delta_{\text{heat}}}
\]

\[
= \frac{\text{latent heat per molecule}}{T^2} \cdot \frac{\text{positive}}{\text{positive}}
\]

\[
= \frac{\text{negative}}{\text{positive}}
\]

\[
\Rightarrow P \propto T^{-2}
\]

Superfluidity in $^3$He a consequence of Cooper pairing, similar to superconductivity.

Refer to Conrath's review in same issue of RMP - calling attention of 4.6 Oe (figure).
condensates:

Bosonic: $^4\text{He}$ - strongly interacting - superfluid \( T_c = 2.1 \text{ K} \)

\( \text{atomic BEC, e.g., Rb} \) - weakly or non-interacting \( T_c \approx 3 \text{ K} \) 

(\text{hard part is density})

Paired Fermions:

usual type-I sc. (e.g., Pb) \( \Delta \)-wave \((l=0)\) \( T_c = 7.2 \text{ K} \) BCS

usual type-II sc. (e.g., \( \text{Nb}_3\text{Sn} \)) \( \Delta \)-wave \((l=0)\) \( T_c = 8.3 \text{ K} \) BCS

heavy Fermion type-II sc. (e.g., \( \text{UPt}_3 \)) \( \Delta \)-wave ? \( T_c = 0.53 \text{ K} \) ?

\( ^3\text{He} \) - A

\( ^3\text{He} \) - B

\( \Phi \)-wave \((l=1)\) \( T_c = 2.5 \text{ K} \) BCS sec p.211

high-\( T_c \) sc. (e.g., \( \text{Tl}_2\text{Ba}_2\text{Cu}_2\text{O}_8 \)) \( \Delta \)-wave ? \((l=2)\) \( T_c = 125 \text{ K} \) ?

The "Bosons" are of course also grouped Fermions - there are two ways
This interpolates...

4/7/21

\( \Phi \)-radiation refs/notes

today: condensates

\( \Delta \)-wave "cooling" stopping - need only give a chance

Wick: kinetic theory

Ingredients in cooling:

2. \( \text{magnetic trap} \) - multi-filament \( \text{uniform} \) \text{schematic} ?

\( \text{optical trapping} \)

\( \text{optical trapping} \) - \text{directional} \text{uniform}

\( \text{directional} \) - \text{schematic}
2. Doppler cooling: "optical molasses"

\[ \kappa, k, b \text{ decay } (\phi) \]

\[ \omega = \omega_{0} - \omega \]

\[ \phi = \pi / 2 \text{ turns out to be optimal} \]

Then we find that applying the same forces with opposite direction slows it down regardless of direction.

\[ \text{Force on atom } \phi = -\mu B \text{ or } \phi = \mu B \text{ (blue shift / red shift)} \]

Resulting velocity distribution nearly Maxwellian, so \( T \) is well-defined.

Doppler \( \rightarrow \sim 20 \text{ mK} \) competition from laser heating \( \rightarrow \text{ random heating} \)

Inverted cavity (cavity cooling)

3. Super-doppler cooling: Stark shift in electric field \( \rightarrow \text{ cool } \)

In magnetic optical trap

\[ \text{Pointed light to conserve angular momentum} \]

\[ \text{incorporates Doppler} \]

4. Superdoppler cooling \( \rightarrow \sim 5 \text{ mK} \)

Return rapidly pol. aligned high-energ. \( \rightarrow \text{ not fast} \text{, slow} \text{, cool atoms} \)

Describe recoil or velocity distribution

\[ \text{attitude demagnetization} \rightarrow \text{ faster} \text{, and} \]
Problem 12.41

At steady state, we're removing heat at rate \( q = 40 \, \text{W} \)

\[ L_0 = \text{latent heat per mole} = 82 \, \text{J/mole} \]

**Table 12.1**

Assumed initial \( \text{Ta} \) \( T \) (or \( T/2 \))

Ignoring any resistance to flow, pressure some until pump clearance is pump in case it exhibits a large, steady state before an equilibrium except for center of maximum flow and then check for pump transfer

Effective pump speed \( \omega' = \frac{\text{number of strokes}}{\text{sec}} \)

\[ \omega' = \omega \cdot \eta \cdot \text{factor} \]

\[ \frac{\text{W}}{\text{m} \cdot \text{sec}} = \frac{\text{kg} \cdot \text{m}^2}{\text{sec}^2} \]

\[ \frac{\text{W}}{\text{m} \cdot \text{sec}} = \frac{\text{kg} \cdot \text{m}^2}{\text{sec}^2} \]

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\[ \frac{\text{W}}{\text{m} \cdot \text{sec}} = \frac{\text{kg} \cdot \text{m}^2}{\text{sec}^2} \]
Classical kinetic derivation of ideal gas law

Local, high density in equilibrium - follows

1. Gaseous concentration \( n \) - set by assuming negligible occupancy

2. Normalizing gaseous \( \rho = \frac{1}{\sqrt{2\pi n}} \)

3. \( \beta = \frac{1}{(2\pi n)^{1/2}} \)

4. \( \alpha = \frac{\sqrt{2} \rho}{I} \)

5. \( \frac{\rho}{I} = \frac{\sqrt{2} \rho}{I} - \frac{\sqrt{2} \rho}{I} \)

6. 1 = \( \frac{(2E)}{(2\pi m)^{1/2}} \)

Kinetically, - collision any elementary process

\begin{align*}
\text{momentum change of wall due to one collision} &= 2mV_1^2 \sin \theta_1
\end{align*}

There are many collisions. Let \( dN \) be the prob. for a particle to have velocity between \( V_1 \) and \( V_1 + dV_1 \)

\begin{align*}
\text{\# per sec} &= \int_{V_1}^{V_2} dN
\end{align*}

\begin{align*}
\text{force} &= \frac{\text{\# per sec}}{\text{time}} \times \text{unit force} = \frac{\text{\# that hit wall}}{\text{unit force}}
\end{align*}

\begin{align*}
\text{pressure} &= \frac{\text{force}}{\text{area}} = \frac{\text{\# that hit wall}}{\text{area} \times \text{unit time}}
\end{align*}

- it is very useful as a tool to think about the situation with \( V_1 \) between \( V_1 \) and \( V_2 \).
\[
\frac{\text{# to hit}}{\text{hit }\:	ext{in the region}} = \int \int \int_{V_2} \frac{V_1}{V_2} \text{ d}V_1 \cdot \text{ d}V_2 \cdot \text{ d}V_3
\]

> pressure due to \( V_2 \) = \( \frac{2\pi \hbar V_2^4}{\pi^2} \cdot \text{d}V_1 \cdot \text{d}V_2 \cdot \text{d}V_3 \)

\[
P = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} 2\pi \hbar V_2^4 \delta(V_1) \delta(V_2) \delta(V_3) \text{ d}V_1 \cdot \text{d}V_2 \cdot \text{d}V_3
\]

\[
\text{since } \delta \text{ is a dirac delta function,}
\]

\[
P = \pi \hbar V_2^4
\]

we have

\[
P = \frac{\pi \hbar V_2^4}{V}
\]

We've checked, since the for our discussion of equipartition has been

quantum mechanical. Outline of proof classical approach (see Fynn, et al.)

1. The dep. of temp. by heat engines

2. When the quasi can come energy, kinetic energy.

3. \( \frac{1}{2} m \langle v^2 \rangle = \frac{1}{2} m \langle v_1^2 + v_2^2 + v_3^2 \rangle \)

Maxwell's velocity distribution

\[
\begin{align*}
\frac{\text{d}N}{\text{d}v} & = N \frac{1}{\sqrt{2\pi \hbar}} \text{exp} \left( -\frac{\hbar \omega}{2\hbar} \right) \\
& = N \frac{1}{\sqrt{2\pi k_B T}} \text{exp} \left( -\frac{\hbar \omega}{2k_B T} \right)
\end{align*}
\]
**Lagrangian derivation:**

\[ P_{\text{Lagrangian}} = \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{(x - \mu)^2}{2\sigma^2}} = P_{\text{Lagrangian}}(x) \]

**so** \[ P_{\text{Lagrangian}} = \frac{1}{\sqrt{2\pi \sigma^2}} 2^{-\frac{1}{2}} \]

**and of** \( x \) \( \gamma \) \( \text{by} \) \( \text{equity} \) \( \text{normalization} \)

\[ PV = Nk_bT \]

**Maxwell velocity distribution**

**body** Diffusion

**For** Boltzmann equation

**Example verification of Maxwell velocity distribution:**

\[ \text{Maxwell velocity distribution} \]

\[ \text{Diffusion} \]

**Mean free path:**

**transporting gas:** molecules move \textit{ballistically} until they collide; afterward, directions are random.

**mean free path** \( \ell \) : average distance between collisions;

\[ \ell = \frac{v}{a} \]

\( v \) : average speed \( \text{of gas} \)

\( a \) : relaxation time \( \text{or mean free time} \)

**Rigid sphere model:**

\[ \text{collide if they're within} \] \( \frac{1}{2} \) \text{ of each other}.

**moving** molecules swept out volume \( \pi r^2 \)

\[ \text{in that} \] \( \text{it will collide with any other molecule with mass} \] \( m \) \text{in this volume}.

\[ \frac{1}{\gamma} \]

\[ \text{next to me,} \] \( \text{other molecules in this volume for} \) \( n \) \text{ not for} \]

**concentration** \( n \) : \( \frac{1}{\gamma} \)

\[ \text{as} \] \( \frac{1}{\gamma} \) \( \text{and} \]

\[ \gamma \]
Linear response: Response to dissipative system is linear to some order.

\[ \text{Response} - \text{Dissipative system is linear to some order.} \]

\[ \text{Field} \]
- \[ \text{Field} \]
- \[ \text{Response} \]
- \[ \text{Heat flux} \]
- \[ \text{Current density} \]
- \[ \text{Molar flow} \]
- \[ \text{Third law} \]

\[ \nabla \cdot \mathbf{j} = \nabla \cdot \mathbf{j} = \rho \cdot \mathbf{v} \]

\[ \nabla \cdot \mathbf{j} = -\rho \cdot \mathbf{v} \]

\[ \text{Start with particle diffusion} \]

\[ n(z) \]

\[ \text{Assume T same on both sides so that } \nabla \cdot \mathbf{j} = 0 \text{.} \]

\[ \text{Flux} \]

\[ \mathbf{j}_n = -D \nabla n \]

\[ \text{Units: } \text{mole}^{-1} \text{cm}^2 \text{ s}^{-1} \]

\[ \text{Flux per unit} \]

\[ \nabla \cdot \mathbf{j} = 0 \text{.} \]

\[ \text{Explanation:} \]
- \[ \nabla \cdot \mathbf{j} = 0 \text{.} \]
- \[ \text{No constraint because } n \text{ is constant.} \]
- \[ \text{No } n, D, \text{ etc. because no difference in } n \text{ or } D \text{ at z.} \]
- \[ \text{\nabla \cdot \mathbf{j} = 0 \text{ but } n \text{ is constant.} \]

\[ \frac{\partial n}{\partial t} = D \frac{\partial^2 n}{\partial z^2} \]

\[ \text{This leads Fick's law.}\]

\[ \text{See also } \text{WHU.
Thin through plane = right moving flux - left moving flux

Right flux: proportional to density \( n \)
- and to a mean speed \( u_e \) in a direction
- factor of \( \frac{1}{2} \) because only \( \frac{1}{2} \) of particles move to right

What is \( u_e \)? Are a proj. of velo. only of those particles that move to the right

- where is density \( n \)? Not at \( u_e \), but in the limit it is \( \frac{u_e}{2} \), proj of \( u_e \). Other density then determines that it is.

\( \frac{u_e}{2} \) \& \( u_e \) are correlated, so it does not make sense to use the value of \( \frac{u_e}{2} \) derived in problem 14.1

\[ \langle u_e \rangle = \langle u_e \rangle \]

\[ \frac{1}{2} \langle u_e \rangle \]

\[ \langle u_e \rangle \neq \langle u_e \rangle \]

- we'll need the latter.

For now leave \( \frac{u_e}{2} \). We'll take the average later.

Right flux = \[ \frac{1}{2} \left( u_e - u_e \right) \]
Left flux = \[ \frac{1}{2} \left( -u_e - u_e \right) \]

Total flux: \[ J_\| = \frac{1}{2} \left[ \langle u_e \rangle - \langle u_e \rangle \right] \cdot \frac{2 \langle u_e \rangle}{2 \langle u_e \rangle} \]

\[ \frac{\langle u_e \rangle}{2 \langle u_e \rangle} \]

Is \( \frac{\langle u_e \rangle}{2 \langle u_e \rangle} \) a key point of interest?

\[ J_\| = - \langle u_e \rangle \frac{\langle u_e \rangle}{2 \langle u_e \rangle} \]

\[ \text{Axis is our only forward direction.} \]

\[ \text{We're mostly concerned - right.} \]

\[ \langle u_e \rangle = \langle u_e \rangle \cos \theta \]

\[ l_2 = l_0 \cos \theta \]

\[ t_0 = \text{relativistic cos} = \frac{v}{c} \]

\[ \langle u_e \rangle = \frac{v}{x} \left( \frac{dA \langle u_e \rangle}{dx} \right) \]

\[ \langle \cdot \rangle \text{mean} = \int \langle u_e \rangle \langle u_e \rangle \]
\[ \mathcal{J}_a = -\frac{1}{2} \frac{\partial^2 \rho}{\partial t^2} = -\theta \frac{J_0}{J_2} \]

so \[ D = \frac{1}{2} \frac{\partial^2 \rho}{\partial t^2} \] in kinetic mode.

More generally,
\[ \mathcal{J} = -D \nabla \rho \]

\[ \begin{bmatrix} \mathcal{J}_a \\ J_2 \end{bmatrix} = -\nabla \cdot \mathcal{J} \quad \text{diffusion equation} \]

\[ = D \nabla^2 \rho \]

constant and equal, with no 1st time derivative.

Similar for thermal diffusion:
\[ \rho(\mathbf{x}) \rightarrow \rho(\mathbf{x}) \text{ energy density (i.e., } \mathbf{c}^2 \mathbf{v}) \]

For solid, energy would be carried by phonons.

Assume instead a gas with density balanced

So that there's no net flow of particles, only of
cycles (more complicated in both particles & heat streams).

To simplify expressions, use \( n^+ = n(2 + k_e) \), \( n^- = n(2 - k_e) \), \( c^+ = c(2 + k_e) \), etc.

\[ \text{Einsteinans to right } = \frac{1}{2} n^+ c^+ \frac{k_B T}{2} \]

\[ \text{to let } = \frac{1}{2} n^+ c^+ \frac{k_B T}{2} \]

\[ \text{particle from right } = \frac{1}{2} n^+ c^+ \text{ since } n^+ = n \]

\[ \text{to let } = \frac{1}{2} n^+ c^+ \]

\[ \text{let every side } \mathcal{J}_2 = \frac{1}{2} n^+ c^+ \frac{k_B T}{2} \frac{dT}{dt} = -n c^+ \frac{1}{2} k_B \frac{dT}{dt} \text{, } \frac{1}{2} k_B \frac{dT}{dt} \]

\[ = -\frac{1}{2} n k_B c^+ \frac{dT}{dt} \]

\[ \mathcal{J}_2 = -\theta \frac{dT}{dt} \]

\[ \text{and } T = \frac{1}{2} k_B c^+ \frac{dT}{dt} - \frac{1}{2} c^+ \]

\[ \text{hence } \frac{d}{dT} = \frac{1}{2} k_B c^+ \frac{1}{2} = \frac{1}{2} k_B c^+ \]
Fri. diffusion

today: diffusion
Boltzmann eqn.

Wed.: modern critical phenomena

From Fri.: \( J_x = -k \nabla T \), \( \n = \frac{1}{3} n_b \langle C \rangle = \frac{1}{3} c \nu C \), \( C_v = \frac{2}{3} \mu \), \( \xi = \langle 1, n \rangle \)

How does \( k \) depend on concentration (equivalently pressure)?

\( C_v \propto n \)
\( \xi \sim \frac{1}{n} \)
result from Fri.

So \( k \) kind of \( n \) (or at fixed \( T \) or \( P \))

Partial vacuum conducts heat exactly as well as room-pressure air: fewer particles to carry heat, but each one moves further before scattering. Effects cancel. This picture breaks down only after a physical situation.

Then, \( k = \frac{1}{3} c \nu C \). As we lower pressure (forcing \( \xi \) to decrease), \( C_v \) (how could it?) but \( C_v \) gets smaller - "viscous regime."

Viscosity:

\[
\frac{N}{\text{field}} \quad \Rightarrow \quad \frac{dy}{dx}
\]

\( F_{\text{friction}} = \mu x \text{ force in } x \text{ direction (opposite).} \)

\[
\int F_y = -\frac{d}{dx}
\]

For a more general treatment, \( J \cdot d \ell \), more complicated fluid behavior.

Result from ideal gases: \( \frac{1}{2} \mu \text{ c}^2 \), \( \rho = \text{ mass density} \)
[\text{\textbf{Equilibrium}}]

\textbf{Boltzmann Eq.:} \textbf{(Really strange how this copy)}

- Concept: distribution function in single-particle semiclassical phase space, \( f = f(\mathbf{r}, \mathbf{p}, t) \) to time
- and its equilibrium value to \( f(\mathbf{r}, \mathbf{p}) \)

Particles have trajectory in \( \mathbf{x}-\mathbf{p} \) space. (1-dim analog: \( \mathbf{x} \))

- In words:

\( f(\mathbf{x}, \mathbf{p}) \) changes due to \textit{collisions} of the \textit{particles}, not \textit{time}.

- Model: relaxation time \( \tau \) (\( \tau \) is a \( \mathbf{x}-\mathbf{p} \) but \( \tau \) called it \( \mathbf{x}\) in some analog)\)

- Two sorts of Things changing:
  1. Particle moves in \( \mathbf{x}-\mathbf{p} \) space: \( \mathbf{r} = \mathbf{r}(t), \mathbf{p} = \mathbf{p}(t) \)
  2. \( f(\mathbf{r}, \mathbf{p}, t) \), may itself depend on \( t \)

Term of driving around an \textit{anticanonical landscape} in \( \mathbf{x}-\mathbf{p} \) valley, etc. changes but shape:

Starting point:

\[
\frac{df}{dt} = \left( \frac{df}{dt} \right)_{\text{collisions}} + \left( \frac{df}{dt} \right)_{\text{time}} + \left( \frac{df}{dt} \right)_{\text{everything else}}
\]

Notes:
1) Subscripts here are not time \( \left( \frac{df}{dt} \right)_{\text{t}} \), but on \( \mathbf{x}-\mathbf{p} \) subset.
2) Last term represents change in \( t \) at fixed point \( \frac{df}{dt} \) in \( \mathbf{x}-\mathbf{p} \) space. We could call it \( \left( \frac{df}{dt} \right)_{t} \). In first derivative, \( \frac{df}{dt} \) (time derivative)
   is removed in the sum of \( \mathbf{x}-\mathbf{p} \), \( \mathbf{r}, \mathbf{p}, t \) (comes from \textit{collisions}, etc.).

\( \text{Ident} \), \( \mathbf{x}-\mathbf{p} \) collision term.

Remark: The approximation: \textit{Fraction} changes in the \( \mathbf{x}-\mathbf{p} \) is \( \frac{1}{2} \) \textit{fraction} change in \( \mathbf{x} \).

\[ \text{To compute fraction change in a non-trivial time } t, \]

\[ \text{write } t' = \text{unit time unit time } \text{and note that, } \]

\[ \text{now, re-arrange } t. \]
Fraction not scattered in time $\Delta t$ is \((1 - \frac{\Delta t}{t_0})\) \((1 - \frac{\Delta t}{t_0}) \cdots (1 - \frac{\Delta t}{t_0})\) for simplicity, we'll assume $t_0$ is at $t$.

\[
\frac{1 - \frac{\Delta t}{t_0}}{t_0} = e^{-\frac{\Delta t}{t_0}}
\]

(eg. by Taylor series)

Consider first a small piece of phase space at equilibrium:

\[
\begin{array}{c}
\text{in} = \frac{dt}{t_0} f_0 \\
\text{out} = \frac{dt}{t_0} f_0 \text{ for equilibrium}
\end{array}
\]

\[
\frac{dt}{t_0} f_0 = \left(\text{fraction scattered}\right) f_0
\]

Now consider non-equilibrium distribution:

\[
\begin{array}{c}
\text{in} = \frac{dt}{t_0} f_0 \\
\text{out} = \frac{dt}{t_0} f(\vec{r}, \vec{p}, t)
\end{array}
\]

Relaxation-time approximation says that collisions cause memory of previous non-equilibrium distribution, so inceasing is given $\frac{dt}{t_0} f_0$ (DDRW).

Note: \[
\left(\frac{\partial}{\partial t}\right)_{\text{coll}} = \frac{1}{t_0} (t_0 - t)
\]

\[
\frac{\partial f}{\partial t} = \text{term in braces or cross terms}
\]

Write \[
\frac{\partial f}{\partial t} = \text{term in braces or cross terms}
\]

\[
\frac{\partial f}{\partial t} = -\frac{1}{t_0} \frac{\partial f}{\partial \vec{r}}
\]

\[
\Rightarrow \quad f(\vec{r}) = f(\vec{r}_0) e^{-\frac{t}{t_0}}
\]
Flow term. Absorb collisions and fields, particle move with constant $\hat{\mathbf{v}}$

\[ \mathbf{f}(\mathbf{r}, \mathbf{p}, t) = \mathbf{f}(\mathbf{r}(t-t_0), \mathbf{p}(t-t_0), t_0) \]

In words: to figure out what's going on at $\mathbf{r}, \mathbf{p}$ at time $t$, look back at $\mathbf{r}(t-t_0), \mathbf{p}(t-t_0)$ at time $t_0$.

Picture:

Note: \[ \mathbf{v} = \frac{\mathbf{p}}{m} \]

Initial term - last $\mathbf{F}$ be force due to external.

Take time derivative.

\[ \left( \frac{\partial}{\partial t} \right)_{\mathbf{r} \mathbf{v}} = -\nabla \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} = \left( \mathbf{V}_\mathbf{v} \cdot \frac{\partial}{\partial t} \mathbf{v} \right) \]

The next sign is confusing - those were studied (and shown to recognize it as the convective derivative).

Final term: let $\mathbf{F}$ be force due to external fields, e.g., $\mathbf{F} = \mathbf{F}(\mathbf{r}, \mathbf{v})$.

Newton's second law: $\mathbf{F} = \frac{d\mathbf{v}}{dt}$

$\mathbf{F}$ change $\mathbf{v}$ directly, $\mathbf{f}$ does not.

\[ \mathbf{f}(\mathbf{r}, \mathbf{p}, t_0) = \mathbf{f}(\mathbf{r}, \mathbf{p} - \frac{\partial \mathbf{v}}{\partial t} t_0, t) + \text{non-collinearity} \]

Pick at will since $\mathbf{f}$ has no effect between $t$ and $t_0$.

\[ \left( \frac{\partial}{\partial t} \right)_{\mathbf{r} \mathbf{v}} = -\nabla \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} = \left( \mathbf{V}_\mathbf{v} \cdot \frac{\partial}{\partial t} \mathbf{v} \right) \]

Putting together the $\mathbf{f}$ terms:

\[ \frac{d\mathbf{v}}{dt} = \frac{\mathbf{F}}{m} - \mathbf{V}_\mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} \mathbf{v} \]
Use the tension to redefine $T = -D \nabla \varepsilon$, $D = \frac{1}{2} \varepsilon$.

No external field if $\varepsilon(\frac{x}{a})$ is zero, so simply take $\varepsilon(\frac{x}{a}) = \frac{\Delta}{2} x^2$.

In steady state $\frac{dx}{dt} = 0$, so Boltzmann reads

$$\frac{1}{\Delta} \frac{dx}{dt} = \frac{x}{\Delta} \frac{d}{dx}$$

$$t = t_0 - v_x \frac{\Delta}{v_x}$$

$$x = x_0 - v_x \frac{\Delta}{v_x}$$

To use a better approximation, say $v_x \approx 0$ to find out the wrong limit

$$t = t_0 - \frac{\Delta}{v_x} \left( \frac{\Delta}{v_x} - v_x \frac{\Delta}{v_x} \right)$$

$$\text{Reversing for using \textit{second} term with \textit{no} \textit{linear} response.}\quad v_x \text{ really small.}$$

$$J = \int \frac{\frac{1}{2} \hbar \hbar^2}{(2\pi)^3} \nu^2 \int f(p,k) \nu_x$$

$$p = \frac{n}{2} \nu = \frac{m}{2}$$

$$e = \int \frac{1}{(2\pi)^3} \frac{1}{\nu_x^2} \int \frac{f(\nu, k)}{\nu_x}$$

$$C \text{ since } f_0 = \frac{1}{2} \hbar^2 \text{ is even in } k, \text{ and } \nu_x = \frac{\hbar}{m} \text{ odd.}$$

To compute second integral, note \( \frac{\hbar^2}{2m} = \hbar k \frac{d}{dx} \frac{d}{dx} \)

Two choices: 1) assume to int. of $k^2$ (i.e., $f \nu^2$) so that it vanishes out.

2) assume $k \sim \hbar \nu^2$, $\nu = \frac{k}{\hbar} \frac{d}{dx} \frac{d}{dx}$.
These two means just differ by \( 0.5\% \). Solid state version #1, width at edges #2.

which is right? Neither — contains non-zero co-variants, no model.

Both next #3 (at least) via phenomenological parameter.

**Part 1.**

Then

\[
J = -\frac{\mu}{f} - 3 \int \frac{\dot{r}^2}{c^2} \to \frac{12L}{(2\pi)^2}
\]

4. \( \frac{\partial \rho}{\partial t} = \frac{1}{2} \left[ \dot{b} \cdot \dot{r} + \frac{\dot{r} \cdot \dot{r}}{c^2} \right] \). Only \( \rho \) depends on \( r \), so

\[
\frac{\partial \rho}{\partial t} = \frac{1}{2} \left[ \dot{b} \cdot \dot{r} \right] = \frac{1}{2} \frac{\partial \rho}{\partial t}
\]

with \( \dot{b} \) in necessary, but elements didn't need in this calculation, since it dropped out.

5. \( \dot{b} \cdot V^2 \to \frac{1}{2} \dot{r}^2 \). Then \( \rho = \frac{1}{2} \dot{b} \cdot \dot{r} = \frac{12L}{(2\pi)^2} \)

\[
J = \frac{1}{2} \dot{b} \cdot \dot{r} = \frac{12L}{(2\pi)^2}
\]

6. \( \nabla \cdot \dot{b} \nabla \dot{r} \). Then \( \nabla \cdot \dot{b} \nabla \dot{r} = \frac{1}{2} \dot{b} \cdot \dot{r} = \frac{12L}{(2\pi)^2}
\]

\[
\nabla \cdot \dot{b} \nabla \dot{r} = \frac{12L}{(2\pi)^2}
\]

4/15/03

M. B. Berman

*term: critical exposure
Fri. reanalysis map


- F. W. Berman, *Critical Exposure, a primer - 1992"
- *J. P. Berman, Lecturer on Phase Transition & the Reanalysis Exps.
- J. F. V. Berman, *Statistical Mechanics of Phase Transitions*
Observations on coexistence line:

1. **Boiling** - two coexisting fluids, even transparent (density, refractive index)

2. **Sunk** - difference in density, even density

4. No discontinuous change in appearance or composition, no latent heat

3. $T < T_c$ - transparent

$T > T_c$ - transparent

$T = T_c$ - critical opalescence:
- Forward scattered light brownish-orange
- "Straw" like a sun on a sunny day

Side scattered: bluish
- Like steam but immiscible straying

Universality: plot $\frac{T}{T_c}$ vs $\frac{n}{n_c}$ for various systems - they all converge near the critical point.

Examples:
- $CO_2$, $N_2$, $V_2$, $O_2$, $H_2O$, $CH_4$, $NH_3$
- Single substance per mol
- Complex, polar fluids
- Complex, non-polar fluids

Theoretical difference:
- Theory predicts a single large imaginary location on phase diagram

Quantitatively:
- $T_{\text{fix}} = \frac{T}{T_c}$. Near $T_c$ very nonlinear
- $\beta = \frac{T}{T_c} - 1$
As shown in the example, in Figure 1 the value $\lambda = 0.339 \text{~cm}^{-1}$, $\lambda = 0.339 \text{~cm}^{-1}$, where $\lambda$ is the wavelength of light.

Universality extends to a very different system, the noisy slider of eq. 11:

At first, velocity decreases until some moment, where $V$ apparently not changing. However, for $t > 0.5t_0$, when $\lambda = 0.25$, the slider at "critical region" will eventually acquire critical region.

Decomposition of $f(x)$:

$$f(x) = x^2 + ax + b$$

In $f(x)$,

$$\frac{df}{dx} = a$$

Graphically, slope at vertex: $f(x)$

Case 1: $f(x) = x^2 + ax + b$

Case 2: $f(x) = x^2 + ax + b$

This also works if there are log or other functions, e.g.,

$$f(x) = x^{1/2} + b$$

$$\frac{df}{dx} = \frac{1}{2}x^{-1/2} + b$$

$$f(x) = x^{1/2} + b$$

$$\frac{df}{dx} = \frac{1}{2}x^{-1/2} + b$$

London Theory for $F$:

$$F = F_0 + \frac{1}{2} g F_0 (b + c)$$

$$\text{spontaneous charge}$$

$$F_0 = g F_0 (b + c) = 0$$

$$g F_0 = 0$$

$$\Delta F = g F_0 = 0$$

$$\frac{\Delta F}{F_0} = \frac{0}{F_0} = 0$$

$$\Delta F = 0$$
Assume $y_2(t)$ can be Taylor expanded about $t=0$, where $y_2$ changes sign:

$$y_2(t) = t + C + \text{negligible}$$

Only odd $x$ in large $y_2$, $x > 0$

So $y_2 = -x$.

\[ t \quad \frac{1}{y_2} \quad T \quad \frac{1}{y_2} \quad s \]

\[ + \frac{1}{t} \quad (-\frac{1}{t}) \quad - \frac{1}{t} \quad \frac{1}{t} \quad + \frac{1}{t} \]

\[ 2s = 2\int_{-\frac{1}{t}}^{\frac{1}{t}} d\frac{1}{t} = \text{const.} \cdot t^{\frac{1}{2}} \]

So $s = \frac{1}{2}$, \text{in agreement with experiment}

Landau theory equiv. to MFT, both ignore spatial fluctuations.

---

Example of critical exponents:

\[ \sigma_x(t) = |t|^{-\nu} \]

\[ \chi = \lim_{H \to 0} \frac{M}{H^\nu} = t^{-\nu} \]

\[ \xi = t^{-\nu} \]

\[ n_{\parallel} - n_{\perp} = |t|^{-\nu} \]

\[ \eta \quad \xi \quad \rho \quad \sigma \quad \lambda \]

\[ 0 \quad 0.11 \quad 1 \quad 1.23 \quad \frac{1}{2} \quad 0.68 \quad \frac{1}{2} \quad 0.32 \]

---

Relation (above below critical point):

\[ x + \frac{1}{\xi} + \frac{1}{\sigma} = \nu - \nu' \]

\[ \nu = 2 - \nu' \]

\[ y = \xi - \lambda \]

\[ y = \eta \]
Wed: Boltzmann to Fluc
Critical phenomena
Tc?

Model system: Ising

\[ \mathcal{H} = -J \sum_{\langle ij \rangle} m_i m_j - H \sum_i m_i - N \mu \]

H - external field, \( N \mu \) - chemical potential.\[\text{and}\]

1D exactly

2D solved exactly by Grassberger 1981

3D phase? NP-complete (solving the 3D Ising problem)

Solved exactly means expression for free energy per spin \( f(T) \), energy \( w(T) \), specific heat, etc., can be computed except for some simple integrals.

All the critical exponents can be calculated from the exact solution.

Phase transition at temp. \( T_c = 2.269J \)

\[ T < T_c : \text{ spontaneous magnetization} \]

\[ T > T_c : \text{ no net magnetization} \]

\[ M \sim |t|^{\beta} \]

For 2D Ising, \( \beta = \frac{1}{4} \) (exact)

Relevant MFT in week 9 (next week):

\[ m = \tanh(\frac{m}{t'}) \quad , \quad t' = \frac{t}{T} \quad , \quad \beta = t + 1 \]
\[ m = \tanh \left( \frac{m}{\xi} \right), \quad \frac{m}{\xi} \text{ is small near critical so long as } t > 0 \]

\[ x \frac{m}{\xi} - \frac{2}{3} \left( \frac{m}{\xi} \right)^2 + \ldots \]

\[ \epsilon' = 1 - \frac{1}{2} \left( \frac{m}{\xi} \right)^2 \]

\[ m = \sqrt{3 \epsilon' \epsilon'' \xi} \]

\[ m = \left[ 3 (1 - \epsilon') \epsilon'' \xi \right]^{1/2} = \left[ 3 \left(1 + (b + t)^2\right) \right]^{1/2} \eta_{c1/2} \]


Why are Landau theory & MFT equivalent, and why both useful?
They ignore spatial fluctuations.

General approach to problems, not all of which can be solved exactly

**The Normalization Group**

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>Not unique</td>
<td>Not unique</td>
<td>Not unique</td>
</tr>
<tr>
<td>(e.g., real phase)</td>
<td>(e.g., real wave function)</td>
<td>(e.g., real wave function)</td>
</tr>
<tr>
<td>(e.g., real symmetry)</td>
<td>(e.g., real symmetry)</td>
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</tr>
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<td>in BBG, although</td>
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<tr>
<td>inspired by It</td>
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</table>

**ASK about configuration at** \( T = 0, T = \infty \)

**Show copies from Y known**

**T < T_c:** before decoupling in any phase structure
- when used with a detailed correlation graph, determines
- phase configuration

**T = T_c:** under K, finite to **F** + finite point
- T > T_c

**T > T_c:** invariant configuration
- Simplicity: leave off brackets terms \((\eta < 0)\)

- Ah-bout then we lose fit
- fixed point.

**Original theorem:**

\[ \eta^2 = \xi \eta_{1,2} = \eta \eta_{2,1} \]

\[ \eta^2 = \xi (\eta_{1,2} + \eta_{2,1}) \]

Partition \( Z = \frac{Z}{\xi} = \frac{Z}{\xi} = Z^1 + N \eta_1 + N \eta_2 \)
\[ \sum_{\{i\}} \prod_{j} \left( 1 + \exp \left[ -\left( \sum_{k \neq j} z_{ij} \right) \right] \right) \]

Now for the essential step.\(^{2}\) Begin the work for \( n = 2 \) where we have

\[ \sum_{\{i\}} \prod_{j} \left( 1 + \exp \left[ -\sum_{k \neq j} z_{ij} \right] \right) \]

\[ \beta'' = -J' \sum_{i} a_{i} \alpha_{i} - H' \sum_{i} \alpha_{i} - \beta' \]

\[ \psi = \frac{\sqrt{2}}{\lambda} \]

"definition" \( k = 2 \)

\[ \beta' \]

\[ \beta'' \]

\[ \beta' \]

\[ \beta'' \]

\[ \beta' \]

where \( J', H', \alpha \) are terms of \( J, H, \alpha \).

(2 ind. eqns.)

\[ G \text{ eqn. doesn't affect any more eqn.} \]

\[ \text{add } b \text{ into } J \text{ only if } b \text{ is not zero.} \]

\[ \text{convergent solution} \]

\[ \lambda = \frac{\mu}{\sqrt{\mu}} \]

\[ \gamma = \frac{\mu}{\sqrt{\mu}} \]

\[ \text{fixed point, } (x', y') = (x, y) \]

\[ \text{accumulation line } b \text{ is } y = ax + \text{constant} \]

- \( (0,0) \): \( x = 0, y = 0 \)
- \( x = \infty, y = \infty \)
- \( \text{no other fixed point} \]

\[ \text{least stable fixed point not interesting.} \]
Since we have \( Z \), we can get the energy per spin (recall \( F = -k_B T \ln Z \)).

Near (0,0) fixed point, let \( t = x^{-1} \) so \( (x, t) \) fixed = \( (0, 0) \).

First \( x' = 4x = 6^2 x \)
\( \varepsilon' = 2\varepsilon = 6\varepsilon \)

the energy = \( f_0 + f_3 \), to smooth through transition
\( f_3 \) singular.

\[
\frac{1}{x} f_3(x', \varepsilon') = \frac{1}{x} f_3(6^2 x, 6^2 \varepsilon)
\]

\[
f_3(x', \varepsilon') = \frac{1}{x} f_3(6^2 x, 6^2 \varepsilon) = \frac{1}{x} f_3(6^2 x, 6^2 \varepsilon)
\]

Since unpinned beam
\( b \) turns at many spins

**Specific heat**

\[
C_v = -T \frac{\partial^2 F}{\partial T^2} = \left( \frac{\partial^2 F}{\partial x^2} \right)_{T=0}
\]

since we don't have \( t \), we \( x \) as substitute.

\[
\frac{\partial^2 F(x, 0)}{\partial x^2} = \frac{1}{x} 6^2 x^2 \varepsilon \frac{\partial^2 F}{\partial x^2}(6^2 x, 0)
\]

\[
\varepsilon_{xy}(x, 0) = 6^2 \varepsilon (6^2 x, 0)
\]

Forgot that \( b = 2 \), and choose \( x \) to be \( x^{-1/2} \). Then

\[
C_v \sim -\frac{\partial^2 F}{\partial x^2} \sim x^{-1/2} F(1, 0) \Rightarrow x = \frac{1}{b}
\]

![Graph of energy vs. temperature](image)
Review Problems

Mandel 3.1 spin-1 paramagnet

\[ N \text{ spins} \]

\[ Z = \sum_{\{s_i\}} \prod_{i} e^{\beta \mu B s_i} = \left[ 1 + 2 \cosh (\beta \mu B) \right]^{N} \]

magnetiz. \( \mathbf{M} = N \mu <s> \); the average spin is the average of any particular spin, so

\[ <s> = \frac{1}{2} \left( 0 - 1 e^{-\beta \mu B} + 1 e^{\beta \mu B} \right) \left[ 1 + 2 \cosh (\beta \mu B) \right]^{-N} \]

\[ = \frac{2 \sinh \left( \beta \mu B \right)}{1 + 2 \cosh \left( \beta \mu B \right)} \]

Small field or high temp. \( \beta \mu B \ll 1 \)

\[ <s> \approx 2 \beta \mu B \]

So \( \mathbf{M} = \frac{2}{3} N \mu^2 \beta \mathbf{B} \)

High field or low temp. \( \beta \mu B \gg 1 \)

\[ <s> = \frac{e^{\beta \mu B} (1 - e^{-2\beta \mu B})}{e^{\beta \mu B} (1 + e^{-2\beta \mu B} + e^{-2\beta \mu B})} \approx 1 - e^{-\beta \mu B} \]

Mandel 4.2

\[ \frac{P}{V} \quad \frac{P}{T} \quad \Rightarrow \quad \frac{P}{V} \quad \frac{P}{T} \]

Volumes are same \( P, T, V \) same

First, what is equilibrium temperature, \( T_e? \)

Energy budget:

\[ A \quad \text{work} \quad \Rightarrow \quad B \quad \text{work} \]
Energy changes (after minus before) from separation:

in A \[ \Delta U_A = \frac{3}{2} Nk_B \left(T_0 - T_A\right) \]
in B \[ \Delta U_B = \frac{3}{2} Nk_B \left(T_0 - T_B\right) \]

The parts of energy changes due to work done on each gas:

to A \[ W_A = P \left(V_A - V\right) = Nk_B \left(T_A - T_0\right) \]
to B \[ W_B = Nk_B \left(T_0 - T_B\right) \]

The difference is heat transferred in to each gas:

\[ Q_A = \frac{5}{2} Nk_B \left(T_0 - T_A\right) \]
\[ Q_B = \frac{5}{2} Nk_B \left(T_0 - T_B\right) \]

Recall \( Q_A = W_A \):
\[ T_0 - T_A = T_B - T_0 \]
\[ T_0 = \frac{T_A + T_B}{2} \]
just average \[ \text{In retrospect obvious} \]

Heat capacity of ideal gas \[ C_P = \frac{5}{2} Nk_B \]
\[ \text{(molar of #rpm)} \]

\[ = T \left(\frac{\partial S}{\partial T}\right)_P \]
\[ \Rightarrow \]

\( \left(\frac{\partial S}{\partial T}\right)_P \) \[ = \frac{\frac{5}{2} Nk_B}{T} \]

\[ \Delta S' = \frac{5}{2} Nk_B \left[ \int_{T_0}^{T_A} \left(\frac{1}{T} dT\right) - \int_{T_0}^{T_B} \left(\frac{1}{T} dT\right) \right] \]
\[ = \frac{5}{2} Nk_B \left(2 \ln T_A - 2 \ln T_0 - 2 \ln T_B\right) \]

Use \( T_0 \approx \text{approx.} \)

\[ \Delta S' = \frac{5}{2} Nk_B \ln \left[ \frac{T_A + T_B}{T_0 T_0} \right] \]

Assertion: the argument of the log is \( \geq 1 \).

Write \( T_0 = T_A + x, \quad T_A = T_0 - x \).
\[ y = \frac{\alpha}{T_0 - T_x} + \frac{\beta}{(T_0 - x)^{\gamma}} \]

Find equation: 
\[ \frac{dy}{dx} = \frac{T_0^\gamma}{(T_0 - x)^{\gamma + 1}} - \frac{\beta T_0^\gamma}{(T_0 - x)^{\gamma + 2}} = 0 \]

\[ T_0 + x = T_0 - x \Rightarrow x = 0 \]

Only extremum

\[ y(x) \to +\infty \text{ at } x = \pm T_0 \]

so \( x = 0 \) is a minimum.

Minimum value is \( \frac{T_0^\gamma}{(T_0)^{\gamma + 1}} = \frac{T_0}{T_0^\gamma} = 1 \)

so \( \Delta y > 0 \) with equality when \( x = 0 \), i.e., \( T_0 = T_x \).

Alternatively, we could have used similar triangles.

Problem: Find the functional form of \( P(V) \) to an ambient of an ideal monatomic gas.

1. Similar triangles \( \tilde{S} = N T_x \frac{(\frac{P}{T_0})^{\frac{n}{n-1}} + 1}{1} \), \( Q = T^{\frac{n}{2}} \), \( n = \frac{N}{\gamma} \)

\[ \Rightarrow T^{\frac{n}{2}} V = \text{constant} \]

i.e., for \( (\tilde{P}, \tilde{T}, \tilde{V}) \) and \( (P, T, V) \) two points on the surface,

\[ \left( \frac{\tilde{T}}{T} \right)^{\frac{n}{2}} \left( \frac{\tilde{V}}{V} \right) = 1 \]

Replace \( T \to \frac{V^2}{N T_0} \)

\[ \left( \frac{\tilde{P}}{P} \right)^{\frac{n}{2}} \left( \frac{\tilde{V}}{V} \right) = 1 \Rightarrow \left( \frac{\tilde{P}}{P} \right) \left( \frac{\tilde{V}}{V} \right)^{\frac{n}{2}} = 1 \]

or \( \tilde{P} \sim V^{-\frac{n}{2}} \)

Alternatively: 
\[ dU = \tilde{T} dS - \tilde{P} dV = -\tilde{P} dV \]

or \( \text{work done} \)

\[ \frac{\partial}{\partial S} = \frac{\partial}{\partial \tilde{T}} \frac{S}{\tilde{T}} = -\frac{\tilde{P}}{\tilde{T}} \tilde{V} \]

\[ \frac{2}{3} k \left( \frac{\tilde{T}}{T_0} \right) = -L \tilde{V} \tilde{X} \tilde{V} \]
Kittel 8.7: "to draw" refers to the motor; the heat \( Q_h \) removed from the refrigerator and the heat \( Q_k \) dumped at room temp may each be larger than the work \( W \) supplied by the motor. (All energy understood to be per unit time, so powers)

\[
\begin{align*}
\text{room} & \quad T_h \\
\uparrow \quad \uparrow \quad \uparrow \\
\text{motor} & \quad T_k \\
\downarrow \quad \uparrow \quad \downarrow \\
\text{work} & \quad W
\end{align*}
\]

\[
\begin{align*}
\frac{Q_k}{T_k} & = \frac{Q_h}{T_h} \\
Q_k & = Q_h + W
\end{align*}
\]

\[
W = Q_h - Q_k = \frac{Q_k (T_h - T_k)}{T_k} \quad \text{if} \quad T_k > T_h/2, \quad \text{as it would typically be for a household refrigerator,} \quad W < Q_k, \quad \text{we can cool from} \quad T_h \quad \text{to} \quad \frac{T_h}{2}.
\]

7.14 N. Bowen, sp10

Good problem to distinguish single-particle from many-body energies.

Single-particle levels \( \epsilon \quad \text{occupied by} \quad \) can be occupied by any number of particles, so any many-body energy integral is possible. We also see that spin does not matter to control \( V \).

\[
\begin{align*}
\frac{f_{BE}(\epsilon)}{f_{BE}(\epsilon')} & = \frac{1}{\epsilon - \epsilon'} \\
\frac{f_{BE}(\epsilon)}{f_{BE}(\epsilon')} & = \frac{1}{\epsilon - \epsilon'} \\
\text{we want} \quad \frac{f_{BE}(\epsilon)}{f_{BE}(\epsilon')} & = 2 = \frac{\epsilon_{BE}(\epsilon)}{\epsilon - \epsilon'}
\end{align*}
\]

At the same time, \( \epsilon \) in \((\epsilon - \epsilon') \) by \( f_{BE}(\epsilon') + f_{BE}(\epsilon') = N' \)

the spin in a sum. \( (\epsilon + \epsilon') \),

\[
\begin{align*}
\left( \frac{f_{BE}(\epsilon)}{f_{BE}(\epsilon')} \right) & = 2 \\
\left( f_{BE}(\epsilon') + f_{BE}(\epsilon') \right) & = N'
\end{align*}
\]
Eliminate $f_{BE}(\xi)$:

$$3 \frac{f_{BE}(\xi)}{n} = 2N.$$  

$$\frac{3}{n} = \frac{1}{N^2 - 1}$$

$$n^{-1} = 1 + \frac{3}{2N}.$$  

Plug in to solve.

$$2 = \frac{e^{\frac{3N}{2}} (1 + \frac{3}{2N}) - \frac{2}{N}}{1 + \frac{3}{2N} - 1} = \frac{2}{3N} \left[ e^{\frac{3N}{2N}} - 1 \right] + e^{\frac{3N}{2N}}$$

$$e^{\frac{3N}{2}} \left( \frac{3N}{2} + 1 \right) - \frac{2}{N} = 2$$

$$\downarrow$$

$$\left( \frac{3N}{2} + 1 \right) = \frac{3N}{2} (1 + \frac{3}{2N})$$

$$\approx 2 \left( 1 + \frac{3N}{2} \right) \frac{1}{2N} \left( 1 - \frac{3}{2N} \right)$$

$$= \frac{3}{2N} \left( \frac{3N}{2} + 1 - \frac{1}{2} \right) + O \left( \frac{1}{N^2} \right)$$

$$= 3 \frac{3}{2N}$$

$$= \frac{9}{2}$$

$$\xi = \frac{9}{2} N + O \left( \frac{1}{N^2} \right)$$  

$$\therefore \xi \gg \sqrt{\frac{N}{2}}$$

**Explanation:**

The result above is not what was expected from an experimental test.

$$\therefore \xi, \xi \ll$$

$$\xi \ll \xi$$

This result is surprising and unexpected.

$$\therefore \xi$$
measuring: 3K
melting-curve thermometer of $^4$He (scent of solid $^3$He) need pressure cell
in thermal but not intimate contact with sample

300K
mercury thermometer
ideal gas ($P V = N kT$)
300K
black-body radiation

producing: 3K
pump on $^4$He
300K
resistive heater, cooling, etc.
300K
resistive heater (cause temp of common tungsten hearthcr bush)
in furnace

Fermi liquid model of Thompson (4.2)
$\Delta$ discontinuity in vortices - liquid

\[ \rho \propto \exp \left( \frac{-\Delta}{kT} \right) \]

\[ C_V = \left( \frac{\partial U}{\partial T} \right)_T \quad ; \quad U_{virial} = \frac{3}{2} N k_b \frac{\Delta}{T} - \frac{N^2 \Delta}{2 V} \quad \text{prob 10.1} \]

$C_V = \frac{3}{2} N k_b$ just as for ideal gas

This is correct for $T > T_c$ but not for $T < T_c$, where we have to follow
liquid and gas branches separately - neither constant volume, constant overall volume

\[ \Delta_H : \quad \Delta V(T>T_c) - \Delta V(T<T_c) = \frac{1}{2} \Delta \]

[Fourier's law is such: 
\[ \phi = \frac{1}{2} \frac{\partial T}{\partial x} \]

\[ \rho \Delta e \tau \]

\[ \rho \Delta e \tau = \frac{1}{2} \frac{\partial T}{\partial x} \]

\[ \rho \Delta e \tau = \frac{1}{2} \frac{\partial T}{\partial x} \]

\[ \rho \Delta e \tau = \frac{1}{2} \frac{\partial T}{\partial x} \]
\( T > T_c \): \( T = 0 \), so \( F = \gamma_0(0) \), \( \gamma_0(T_c^+) = -\frac{1}{T_c} \gamma_0(0) \)

\( T < T_c \): From refs pp.52-53, \( \gamma^2 = \frac{2k}{\bar{y}^4} \) giving min \( \pm 5 \) for \( t \to 0 \)

\[ F = \gamma_0(0) - \frac{1}{2} \frac{\omega}{\bar{y}^4} + \frac{1}{4} \frac{(\omega \bar{y}^2)}{\bar{y}^4} = \gamma_0(0) - \frac{1}{2} \frac{\omega}{\bar{y}^4} \]

\[ \gamma = -\frac{1}{12} \bar{y} \frac{\partial}{\partial \bar{y}} - \frac{\omega}{\bar{y}^4} \]

discontinuity \( \Delta \bar{y} = \frac{1}{12} \frac{\omega}{\bar{y}^4} \)

Some themes

Statistical Mechanics

Ensembles

- \( \text{Entropy} = \frac{k_B}{\bar{y}} ln (\# \text{ configs}) \)
- \( \text{Temp} = \left( \frac{\partial S}{\partial V} \right)_{T, V} \)
- partition function, Boltzmann factor
- quantum of phase space
- Smaiz-Fejde
- Monte Carlo
- fluids, small except near 2nd order critical

Thermodynamics

1. \( \delta S = \delta Q / T \)
2. \( dS > 0 \)
3. \( T > 0, N > 0 \) (usually)

\( \delta T \)

Thermodynamic definition of temp.

potentials (Fn, H, etc.)

Further details

Questions

What are Fermi gas? What metal Fermi gas? Intermolecular potential in gas of common
molecules? What is macroscopic ground state of an ideal gas?