Stacking quasicrystallographic lattices

N. David Mermin and David A. Rabson
Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, New York 14853-2501

Daniel S. Rohrsar
Department of Physics, University of California, Berkeley, Berkeley, California 94720

David C. Wright
Center for Science and International Affairs, Kennedy School of Government, Harvard University,
Cambridge, Massachusetts 02138
(Received 8 January 1990)

Three-dimensional axial quasicrystallographic reciprocal lattices with n-fold symmetry are classified. The classification is complete for general n whenever the horizontal planes contain standard two-dimensional quasicrystallographic sublattices.

I. INTRODUCTION

The discovery of alloys with crystallographically forbidden long-range positional order has inspired a reexamination of the basic crystallographic concepts of lattice and space group from the quasicrystallographic point of view. One extends the celebrated classification of three-dimensional lattices by Frankenstein and Bravais to the quasicrystallographic case by noting that if a lattice is defined in k space as the closure under addition of the set of wave vectors determined by a Bragg-like diffraction pattern, then there is no reason to require a minimum distance between lattice points and therefore no basis for excluding axes of n-fold rotational symmetry for general n. The lattices we examine here are always such “reciprocal” lattices.

From the broader quasicrystallographic perspective, the 14 three-dimensional crystallographic Bravais lattices are an idiosyncratic lot, ten of them being exceptional cases. The three cubic and four orthorhombic lattices have more than a single axis of highest symmetry (as do the three quasicrystallographic icosahedral lattices), while the triclinic and the two monoclinic lattices share with the orthorhombic the other pathology of having no axes of rotational symmetry greater than twofold. Only the four remaining crystallographic Bravais lattices fit into the general quasicrystallographic paradigm, having a unique axis of highest rotational symmetry, which is at least threefold. Such lattices are called axial, and are most simply viewed as stackings of their two-dimensional sublattices.

There are two ways to stack square two-dimensional Bravais lattices with fourfold rotational symmetry to produce three-dimensional axial Bravais lattices.

1. There is a vertical stacking in which they are directly above one another, which preserves the fourfold symmetry, yielding the three-dimensional simple tetragonal lattice.

2. There is a staggered stacking, with an appropriate horizontal shift, yielding the three-dimensional centered tetragonal lattice, which also has fourfold symmetry but repeats in the vertical direction only every other layer.

There are also two ways to stack triangular two-dimensional Bravais lattices with sixfold rotational symmetry into axial lattices.

1. The vertical stacking preserves the sixfold symmetry, yielding the three-dimensional hexagonal lattice.

2. There is a staggered stacking with a horizontal shift, yielding the three-dimensional trigonal lattice, which only has threefold symmetry and repeats in the vertical direction every three layers.

To classify three-dimensional quasicrystals with n-fold axial symmetry for general n, one must determine the number of distinct ways to stack two-dimensional N-fold symmetric lattices. Since the rotational symmetry N of a two-dimensional lattice is necessarily even (lattices are inversion symmetric and inversion is equivalent to a rotation by π in two dimensions), it follows that the smallest value N can have is n or 2n depending on whether the rotational symmetry n of the three-dimensional axial lattice is even or odd. In what follows we do not consider the possibility of stacking two-dimensional lattices with higher values of N to form an n-fold symmetric lattice in three dimensions—i.e., we assume that the stacking is not so skewed that the symmetry of the two-dimensional layers is no longer a consequence of the symmetry of the resulting three-dimensional structure. In the crystallographic case this proviso rules out, for example, the stacking of two-dimensional square lattices into a three-dimensional monoclinic or triclinic lattice, because the preservation of fourfold symmetry in the planes could only be an accident, in view of its absence in the vertical neighborhood of those planes.

In generalizing the categories of stackings of fourfold and sixfold crystallographic lattices to the N-fold quasicrystallographic case, it is useful to represent as complex numbers the vectors of the two-dimensional lattices with N-fold symmetry that constitute the layers of the stack-
ing. With a suitable choice of scale any two-dimensional \( N \)-fold symmetric lattice can be taken to be a set of integral linear combinations of the \( N \)th roots of unity. Rotation by \( 2\pi /N \) about the \( z \) axis perpendicular to the plane then simply corresponds to multiplication by \( e^{2\pi i/N} \). The set of all integral linear combinations of the \( N \)th roots of unity is the most obvious example of a two-dimensional \( N \)-fold symmetric lattice. This set is known to number theorists as \( Z_N \), the cyclotomic integers of degree \( N \); we call such lattices "standard" \( N \)-lattices. The square and triangular lattices are the standard 4- and 6-lattices. Up to a uniform scale factor it turns out that any two-dimensional lattice with \( N \)-fold symmetry is identical to the standard \( N \)-lattice \( Z_N \) when \( N \) is less than 46, but nonstandard lattices abound for larger values of \( N \). We limit our analysis here to the stacking of standard lattices, so our classification of axial quasicrystallographic lattices is complete only when \( n < 23 \).

We specify a stacking of standard lattices by defining the vector \( z \) to be the perpendicular displacement between neighboring layers of the stacking, and the shift vector \( \alpha \) to be the complex number giving the parallel displacement from layer to layer. The stacked three-dimensional lattice consists of all integral linear combinations of the \( N \)th roots of unity and the vector \( z + \alpha \). Note that \( \alpha \) is defined only up to a cyclotomic integer, since shifting the standard lattice by a cyclotomic integer leaves it unchanged. We call a stacking vertical if \( \alpha \) is zero (or a cyclotomic integer) and staggered if it is not. Structures have been reported whose diffraction patterns correspond to reciprocal lattices that are vertical stackings of eightfold, tenfold, and twelvefold two-dimensional lattices.\(^7\)

In classifying lattices one considers two to be equivalent if it is possible to map one onto the other by a continuous family of deformations that preserve the point-group symmetry. In the case of \( n \)-fold symmetric axial lattices with \( n \) greater than two it is enough to consider deformations composed of a rotation, an isotropic horizontal rescaling, and a vertical expansion or compression.\(^8\) (We have already used the freedom to choose a convenient horizontal scale in representing a two-dimensional standard lattice as the set of all cyclotomic integers.)

Unlike crystallographic lattices, standard quasicrystallographic lattices have a symmetry under changes of scale, which plays an important role in their classification. These lattices are left unchanged by multiplication by certain complex numbers, i.e., by a dilation or compression combined with a rotation. The special complex numbers are those cyclotomic integers whose multiplicative inverses are also cyclotomic integers, known to mathematicians as cyclotomic units. It is easy to see the invariance of the cyclotomic integers under such a multiplication. Let \( \mu \) be a unit, so that its inverse \( \lambda \) is also a cyclotomic integer. Since \( \mu \) is a cyclotomic integer, multiplying the cyclotomic integers by \( \mu \) gives nothing but cyclotomic integers. Furthermore, it gives all of them, for any cyclotomic integer \( \gamma \) is given by multiplying the cyclotomic integer \( \lambda \gamma \) by \( \mu \).

An intermediate consequence of this scale invariance of the standard lattice is that two shift vectors whose ratio is a cyclotomic unit give equivalent lattices. One of the primary results of the analysis below is that this makes all staggered stackings with the same rotational symmetry equivalent.\(^9\)

By appealing to special number-theoretic properties of the golden mean, it has been shown\(^2\) that there is just one staggered stacking of a tenfold quasicrystallographic lattice, which repeats every five layers and has fivefold symmetry (following the pattern of the sixfold crystallographic example). By appealing to results from five-dimensional crystallography it has also been shown\(^10\) that there is just one staggered stacking of an eightfold lattice, which repeats every other layer and has eightfold symmetry (following the pattern of the fourfold crystallographic example). We show below that the question of stacking standard \( N \)-lattices has a simple answer\(^11\) for general \( N \);

If \( N \) is a power of 2 (and \( N > 2 \)), then in addition to the vertical stacking there is a unique staggered stacking that repeats vertically every other layer with rotational symmetry \( n = N \), as in the crystallographic case \( N = 4 \).

If \( N/2 \) is a power of an odd prime number \( p \), then in addition to the vertical stacking there is a unique staggered stacking with rotational symmetry \( n = N/2 \) that repeats vertically every \( p \) layers, as in the crystallographic case \( N = 6 \).

If \( N/2 \) has a least two distinct prime factors (which corresponds to no crystallographic case), then there is only the vertical stacking, with rotational symmetry \( n = N \).

These results are summarized in Table I, which reorganizes them in terms of the rotational symmetry \( n \) of the three-dimensional stacked lattices. Vertical stackings are possible only with even \( n \), since they have the same rota-

\[
\begin{array}{|c|c|c|}
\hline
n & Stacking & Crystallographic \text{ ancestry} \\
\hline
n=2^k & Vertical and \text{ Alternating} & \text{Tetragonal} \\
\hline
n \text{ even, } \text{ } n \neq 2^k & \text{Vertical} & \text{Hexagonal} \\
\hline
n \text{ odd, } \text{ } n \neq p^k & \text{Repeats after } p \text{ layers} & \text{Trigonal} \\
\hline
n = p^k & N = 2n & n = 3, 5, 7, 9, 11, 13, \ldots \\
\hline
n \text{ odd, } \text{ } n \neq p^k & \text{None} & \text{None} \\
\hline
(n = 15, 21, 33, 35, \ldots) \\
\hline
\end{array}
\]

TABLE I. Three-dimensional \( n \)-fold axial lattices (\( n \geq 3 \)) with standard \( N \)-fold two-dimensional sublattices. There are no vertical stackings with odd \( n \) since the rotational symmetry \( n \) of a vertical stacking is identical to the rotational symmetry \( N \) of its two-dimensional layers, which is necessarily even. The stacked lattices of odd rotational symmetry are thus necessarily staggered, so they can exist only when \( n \) is a power of an odd prime. Each of the three types of stackings is a family that starts with a single crystallographic ancestor.
tional symmetry $N$ as their two-dimensional layers, and $N$ is necessarily even. A stacked lattice can thus have odd rotational symmetry $n$ only if it is a staggered stacking of $N$-fold two-dimensional lattices with $N = 2n$, and therefore only if $n$ is a power of an odd prime number.

Since standard lattices are the only quasicrystallographic lattices when $N$ is less than 46, this catalog of axial three-dimensional lattices is complete for all $n$ less than 23.

II. THE PROOF

A. Vertical stackings

It is revealed that the stacking $must$ be vertical unless the rotational symmetry $n$ of the three-dimensional stacked lattice is a power of a single prime number. For all the choices of $n$, so that $n = ab$ the stacking must then repeat after $a$ layers, since if $R$ is the rotation about the $z$ axis through $2\pi/n$ and if $z + \beta$ is a vector of the three-dimensional lattice (with $\beta$ a horizontal vector), then

$$(R^b + R^{2b} + \cdots + R^{ab})(z + \beta) = a z,$$

so that $az$ is a vector in the stacked lattice. It follows that if $n$ has two distinct prime factors $p$ and $p'$, then $pz$ and $p'z$ are in the stacked lattice. Since for two distinct primes there are integers $m$ and $m'$ satisfying $mp - m'p' = 1$, $z$ itself is in the stacked lattice and the stacking is vertical.

Thus staggered stackings can arise only when $n$ is a power of a single prime number, $n = p^i$, and such staggered stackings must repeat after $p$ layers. When $n$ has two or more distinct prime factors, the only three-dimensional axial lattices are vertical stackings of two-dimensional $n$-fold sublattices.

We next show that when $n$ is a power of a single prime number, the rotational symmetry of the stacked lattice reduces the number of staggered stackings to $p - 1$ possible forms and the scale invariance of the two-dimensional sublattice makes all $p - 1$ forms equivalent, so there is just one staggered stacking.

B. Rotational symmetry

Since the three-dimensional lattice is invariant under a rotation through $2\pi/n$, so is every layer of the stacking. If we take $z$ to be $e^{2\pi i/n}$, then since $z + \alpha$ is in the second layer, $z + \xi z$ must also be, and therefore their difference, $(\xi - 1)\alpha$, must be in the standard lattice that constitutes the first layer. We therefore have:

$$(\xi - 1)\alpha = \sum_{j=0}^{v-1} m_j \xi^j,$$

Here $v$ is the number of integrally independent $n$th roots of unity when $n = p^i$:

$$v = q(p - 1), \quad q = p^{i-1}.$$  

It follows from (2) that

$$(\xi - 1)\alpha - m = \sum_{j=0}^{v-1} m_j (\xi^j - 1),$$

where $m = m_0 + m_1 + \cdots + m_{v-1}$. Since the ratio

$$\frac{\xi^j - 1}{\xi - 1} = 1 + \xi + \cdots + \xi^{j-1}$$

is a cyclotomic integer, we conclude that $\alpha$ can differ from a cyclotomic integer only by an integral multiple of $1/(\xi - 1)$:

$$\alpha \equiv m \mod (\xi - 1), \quad m = 1, 2, \ldots, p - 1,$$

where $\equiv$ denotes equality to within an additive cyclotomic integer.

We have limited the choices of $m$ to the $p - 1$ specified in (6) because $p \mod (\xi - 1)$ is a cyclotomic integer. This follows from the identity

$$p = (\xi - 1)(1 + \xi + \cdots + \xi^{p-1}) \sum_{j=1}^{p-1} j \xi^{u/j},$$

which follows, in turn, from the fact that if $z \neq 1$ is any $u$th root of unity, then

$$\sum_{j=1}^{u} z^j = 0,$$

as an immediate consequence of which

$$u = (z - 1) \sum_{j=1}^{u-1} j z^j.$$

Applying (9) to the case $z = \xi^d$, $u = p$, and factoring $\xi^d - 1$ gives (7). There are thus at most $p - 1$ inequivalent forms (6) for the shift vector $\alpha$.

Note that for $1 \leq m \leq p - 1$, Eqs. (6) and (7) together express these $p - 1$ choices for $\alpha$ as explicit linear combinations of $v$ angularly consecutive $n$th roots of unity with nonintegral coefficients:

$$\alpha \equiv \frac{m}{p} (1 + \xi + \cdots + \xi^{p-1}) \sum_{j=1}^{p-1} j \xi^{u/j}.$$  

Since the roots of unity appearing in this expansion constitute an integrally independent set, it follows that these $p - 1$ shift vectors cannot be cyclotomic integers, and the $p - 1$ stackings that they give are indeed all staggered.

C. Scale invariance

To show the equivalence of these $p - 1$ staggered stackings we first show that to within additive cyclotomic integers the distinct choices (6) for the shift vector $\alpha$ can alternatively be taken to be

$$\alpha \equiv \frac{1}{\xi^k - 1}, \quad k = 1, 2, \ldots, p - 1$$

(where the $k$ corresponding to a given $m$ in (6) is that for which $mk$ is congruent to 1, modulo $p$).

To establish this correspondence, note that using (9) (with $z = \xi, u = n$), we may reexpress the form (6) for $\alpha$ as
\[
\alpha = m \frac{n}{\xi - 1} = m \sum_{j=1}^{n-1} j \xi^j.
\]  
(12)

For any integer \(w\) we define \([w]\) to be the integer between 1 and \(n\) congruent to \(w\) modulo \(n\). If \(k\) is any integer between 1 and \(n - 1\) not divisible by \(p\), then since the only divisors of \(n\) are powers of \(p\), if \(j\) assumes all integral values from 1 to \(n - 1\), we will \cite{16} \([kj]\). We can therefore rewrite (12) as

\[
\alpha = \frac{mk}{n} \sum_{j=1}^{n-1} j \xi^{kj} = \frac{mk}{n} \sum_{j=1}^{n-1} j \xi^{kj},
\]  
(13)

To within a cyclotomic integer, we can replace \([kj]\) with \(kj\) itself, since they differ by an integral multiple of \(n\) (and since \(\xi^n = 1\)):

\[
\alpha = \frac{mk}{n} \sum_{j=1}^{n-1} j \xi^{kj},
\]  
(14)

or, again invoking (9) (with \(z = \xi^k\) and \(u = n\)),

\[
\alpha = \frac{[mk]}{\xi^k - 1}.
\]  
(15)

But since \(m\) is not divisible by \(p\), as \(k\) runs through all the integers between 1 and \(n - 1\) not divisible by \(p\), so will \cite{17} \([mk]\), and there will therefore be a value of \(k\) for which \([mk]=1\). Furthermore, if we consider just those values \(m'\) that differ from \(m\) by multiples of \(p\) and therefore give equivalent shift vectors \(\alpha\), the corresponding \(k'\) with \([m'k']=1\) must also differ by multiples of \(p\), and therefore one of them will necessarily be in the range between 1 and \(p - 1\).

We conclude that for \(m = 1, 2, \ldots, p - 1\),

\[
\alpha = \frac{m}{\xi - 1} = \frac{1}{\xi^k - 1},
\]  
(16)

where \(k\) can simply be taken as the unique integer between 1 and \(p - 1\) with \(mk\) congruent to 1 modulo \(p\).

The utility of this expression for the \(p - 1\) shift vectors is that it now follows immediately that any two shift vectors in this alternative form differ by a scale factor that is a cyclotomic unit. For consider two shift vectors \(\alpha\) and \(\alpha'\) given by \(1/(\xi^k - 1)\) and \(1/(\xi^{k'} - 1)\). Since neither \(k\) nor \(k'\) is divisible by \(p\), there is an integer \(r\) such that \(rk\) is congruent to \(k'\) modulo \(n\), so that \(\xi^{rk} = \xi^{k'}\). The ratio

\[
\alpha' = \frac{\xi^{rk}}{\xi^{r(k') - 1}} = \frac{\xi^{rk}}{\xi^{rk'}} = 1 + \xi^k + \xi^{2k} + \ldots + \xi^{(r-1)k}
\]  
(17)

is thus a cyclotomic integer \(\mu\). Precisely the same argument with \(k\) and \(k'\) interchanged establishes that \(\alpha' / \alpha\) is also a cyclotomic integer, \(\lambda\). Since the product of these two ratios is unity, the cyclotomic integers \(\lambda\) and \(\mu\) are cyclotomic units.

Thus a stack of standard lattices shifted by \(\alpha\) differs from a stack shifted by \(\alpha'\) only by a rotation through the phase of \(\mu\) and a horizontal rescaling by the magnitude of \(\mu\), and the axial lattices with shift vectors \(\alpha\) and \(\alpha'\) are equivalent.

### III. POINT GROUPS OF THE STACKED LATTICES

The point groups of the vertical stackings with \(n\)-fold rotational symmetry are evidently \(D_{nh} = n/mmm\). It is also easy to show that the point groups of the staggered stackings are \(D_{nh}\) if \(n\) is a power of 2, and \(D_{nd} = n2/m\) if \(N\) is an odd prime power. Since any three-dimensional lattice is inversion symmetric, it suffices to establish the existence of a horizontal twofold axis. The real axis has this property:

A rotation through \(\pi\) about \(\xi^0 = 1\) takes any vector \(z + \beta\) into \(-z + \beta^*\). But the shift vector \(\alpha\) for any layer of the stacking is just the negative of the shift vector for its mirror image in the \(z = 0\) plane. Therefore the lattice will be invariant under this twofold rotation provided the negative of \(\alpha\) differs from its complex conjugate by a vector of the standard two-dimensional sublattice. Since the shift vector for any layer is equivalent to one of the forms (16), we need only note that

\[
-\alpha - \alpha^* = -\frac{1}{\xi^k - 1} - \frac{1}{\xi^{k-1}} = 1 = \xi^0.
\]  
(18)

### IV. CONCLUSIONS

Our primary conclusions are contained in Table 1, but in addition we make two remarks on the merit of doing three-dimensional quasicrystallography in three dimensions, rather than deriving it by projection from the crystallography of higher dimensional spaces:

1. If one insists on considering a lattice to exist in "real space" as well as in Fourier space, then it is necessary to regard the three-dimensional quasicrystallographic lattices as three-dimensional projections of suitably chosen crystallographic lattices in a higher dimension. The dimension required becomes arbitrarily large with the order \(n\) of the symmetry axis. If one wishes to derive by projection the results obtained here directly in three-dimensional reciprocal space for general \(n\), then one requires some knowledge of higher-dimensional crystallography in arbitrarily many dimensions.

2. By regarding three-dimensional crystallography as merely a special case of three-dimensional quasicrystallography, the nonpathological crystallographic categories (of a character heretofore obscured by the preponderance of pathological cases) acquire a new coherence when viewed as the bottoms of towers of larger quasicrystallographic categories. We have demonstrated this elsewhere \cite{18} for the two-dimensional quasicrystallographic space groups with standard lattices, and a perusal of Table 1 demonstrates the same point for the three-dimensional axial quasicrystallographic lattices.

**ACKNOWLEDGMENTS**

This work was supported in part by the National Science Foundation, Grant No. DMR-86-13368. One of us (D.A.R.) received additional support through the Cornell University Mathematical Sciences Institute.
APPENDIX

If $\nu$ is the maximum number of integrally independent cyclotomic integers of degree $n$, then

$$\zeta^k, \zeta^k+1, \ldots, \zeta^k+\nu-1$$

is an integrally independent set. To establish this it suffices to consider the set $1, \zeta, \ldots, \zeta^{\nu-1}$, since the original set differs from it by no more than a rotation. Note first that if there is no vanishing integral linear combination of vectors from a finite set there is also no vanishing rational linear combination (since if there were it could be converted to an integral one by multiplying by the common denominator of all the rational coefficients). Suppose that

$$\sum_{j=0}^{\nu-1} m_j \zeta^j = 0,$$  \hspace{1cm} (A1)

with integral coefficients $m_j$ at least some of which are nonzero. Let $k$ be the largest value of $j$ for which $m_j$ is nonzero. One can then express $\zeta^k$ as a rational linear combination of a set of no more than $\nu-1$ roots of unity, $1, \zeta, \ldots, \zeta^{k-1}$. Multiplying that rational relation by $\zeta$ and (if necessary) using the original relation to eliminate $\zeta^k$ we can express $\zeta^{k+1}$ as a rational linear combination of $1, \zeta, \ldots, \zeta^{k-1}$. By repeating this procedure, we can express all the higher powers of $\zeta$ as rational linear combinations of $1, \zeta, \ldots, \zeta^{k-1}$. Since $k < \nu$, this contradicts the fact that there can be as many as $\nu$ rationally independent cyclotomic integers.

---


3Twofold axes are pathological from the quasicrystallographic point of view because the standard two-dimensional lattice with $N$-fold symmetry (defined below) degenerates to a one-dimensional lattice when $N = 2$.

4It is not obvious that such relaxation to a configuration of lower planar symmetry will necessarily take place in the analogous quasicrystalline case. There are, however, grounds for viewing not as quasicrystals but as incommensurate modulated quasicrystals such peculiar structures as 36-fold standard lattices in a stacking with only ninefold symmetry. We do not pursue these issues here beyond warning the reader that they are not entirely resolved.

5Note that $\zeta_0$ and $\zeta_1$ are identical. In fact, whenever $N$ is twice an odd integer, the cyclotomic integers of degree $N$ are identical to those of degree $N/2$. In this case the convention of the mathematicians is to take the degree to be $N/2$.

6For these and other basic facts about quasicrystallographic lattices, see N. D. Mermin, D. S. Rokhsar, and D. C. Wright, Phys. Rev. Lett. 58, 2099 (1987), and also Ref. 2.


8Three-dimensional crystallographic lattices with only twofold symmetry axes can, in addition, suffer independent deformations in two independent directions perpendicular to the axes. Such lattices do not fit into the scheme developed here because their two-dimensional sublattices are not standard lattices. Although pathological, these low symmetry lattices are crystallographic and therefore well understood.

9The point is simply illustrated in the crystallographic case $n = 3$, where the trigonal lattice is constructed by shifting the triangular lattice from layer to layer by $\frac{1}{3}$ of an appropriate horizontal lattice vector. One might wonder why one does not get a second inequivalent stacking by shifting by $\frac{1}{3}$ of that vector, but a moment’s thought reveals that this merely gives back the first stacking, rotated by $\pi$—i.e., rescaled by the factor $-1$. In the quasicrystallographic case the corresponding factors contain rescalings by the magnitudes of units as well as rotations.


11Stated without proof in Ref. 1.

12These conclusions exploit only the rotational symmetry of the lattice. They are not restricted to standard sublattices, and are therefore valid for general axial lattices.

13That $\phi_0, \phi_1, \ldots, \phi^{n-1}$ constitute an integrally independent set is shown in the Appendix.

14For general $n$ the number of integrally independent $n$th roots of unity is given by the Euler totient function $\phi(n)$, the number of integers less than $n$ that are relatively prime to $n$. If $p_1, p_2, \ldots, p_r$ are the distinct prime numbers that divide $n$, then $\phi(n)$ is most conveniently evaluated as $\phi(n) = (1 - 1/p_1)(1 - 1/p_2)\cdots(1 - 1/p_r)n$.

15See the Appendix.

16Otherwise there would be two values, $j$ and $j'$, less than $n$ such that $k(j - j')$ was a multiple of $n$. But this would require $j - j'$ to be a multiple of $n$, since $k$ is relatively prime to $n$.

17For essentially the same reasons as given in Ref. 16.